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RM-6152-PR
OCTOBER 1969

A PENETRATION GAME MODEL WITH HOMING BUT NO COUNTING FOR THE DEFENSE

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PREFACE

This Memorandum presents the game theoretical solution to a penetration problem involving a force of (attackers) planes or missiles, that pass through a defense position. The force is assumed to be divided into several groups of different sizes. The question of how valuable is it to the offense to have counter-measures which prevent the defense from counting the number of attackers in each group is investigated.

SUMMARY

An attacker has a force of bombers or missiles which he is trying to get past a defense position defended by homing missiles; he can split up his force into one or more groups of different weights. The attacker's countermeasures prevent the defense from counting the number of units in each group. The defense can salvo up to k times at a given group with shoot-look-shoot tactics. Total attack and defense size are known to both sides.

A game theoretic solution is presented for the case of $k = 1$, with one defense salvo per group. The case of $k > 1$ is discussed and solved for several small examples.

CONTENTS

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1. INTRODUCTION AND BACKGROUND

This model was suggested to the author by John Mallett several years ago in connection with a continental air defense study. The game was solved at that time but was never published because the technical picture changed so that it seemed that the model was no longer applicable. Recently, however, it was decided to write up the solution on the grounds that the problem solved may be of sufficient interest in itself. Also, the technical situation could possibly shift back to the original one, making the model and its solution pertinent again.

Briefly, the attack can split up his force (in the original case they were bombers) into separate groups of different sizes to penetrate a defense zone defended by a force of homing missiles. Each missile can kill one bomber in the group. The key assumption is that, due to countermeasures, the defense can tell only that there is at least one bomber left in the group. Thus counting is not possible for the defense, but homing is.

Due to bomber speed, there is only time for k defense salvos against the attacking group before complicated shoot-look-shoot tactics begin. The defense can see only one attacking group at a time. The defense wishes to maximize the number of bombers killed. The attacker wants to minimize the quantity.

In Secs. 2, 3, 4, 5 the above model is solved in general for the case of $k = 1$, with one defense salvo per group. The case where $k > 1$ is discussed briefly and small examples are worked out in Sec. 6 to illustrate the new complications in the solution.

2. MATHEMATICAL FORMULATION OF THE GAME FOR $k = 1$

Let

R = the number of bombers which Red (the attacker) sends past the Blue (defense) position.

r_i = the number of bombers in the i-th group.

$\mathcal{R}(p) = \{r_1, r_2, \ldots, r_p\}$; a Red strategy partitioning R into p groups, where $\sum_1^p r_i = R$ and r_i is a positive integer.

B = the number of homing missiles defending the Blue region.

$\mathcal{B}(q)$ = a Blue strategy partitioning B into q groups.

b_i = the number of missiles assigned to defend against the i-th Red group.

$\mathcal{B}(q) = \{b_1, b_2, \ldots, b_q\}$, a Blue strategy partitioning B into q groups, where $\sum_1^q b_i = B$, and b_i is a positive integer.

$\min(b_i, r_i)$ = the number of kills in the i-th group.

The payoff to Blue is

$$M(\mathcal{B}(q), \mathcal{R}(p)) = \sum_{i=1}^{p} \min(b_i, r_i).$$

Optimal strategies will be exhibited for both sides and the verification carried out. Note that Blue plays nonsequentially, i.e., he does not change plans as the game progresses. The nature of Red's optimal strategy

will make it apparent that Blue cannot take advantage of any sequential information he learns after an encounter with each Red group. Thus he cannot gain by playing sequentially. It will be shown that Blue plays independently of R, while Red needs to know B only within a certain range.

3. OPTIMAL DEFENSE STRATEGY

Blue mixes equally over g integer partitions of B. Groups are formed in such a way that if B_i is the expected number of missiles assigned to the i-th Red group, then

(1) $$B_i = 1 + B_{i+1}, \quad \sum_1^g B_i = B.$$

The value of g is given by

(2) $$\frac{(g-1)g}{2} < B \leq \frac{g(g+1)}{2}.$$

In fact, g is the largest integer $< \frac{1 + \sqrt{1 + 8B}}{2}$. Thus B_1 is found by

(3) $$B = gB_1 - \frac{(g-1)g}{2}.$$

In addition, Blue mixes over the closest integers to B_i for each i, in case B_i is not an integer. Let $\beta_o = \beta_o(g)$ = Blue's optimal strategy. Then β_o can be constructed as follows: First, define the vector G where

$$G = (g-1,\ g-2,\ \ldots,\ 2,\ 1,\ 0).$$

If $B_1 = g$, then $\beta_o(g) = (g, \ldots, 2, 1)$. Otherwise, when B_1 is not an integer, let

(4) $$t = B - \frac{g(g-1)}{2}.$$

Let T(g) be a g × g matrix of entries $t_{ij} = 0$ or 1 such that

$$\sum_{i=1}^{g} t_{ij} = t = \sum_{j=1}^{g} t_{ij}.$$

A simple way to construct $T(g)$ is as follows: Let $t_{1j} = 1$ for $1 \leq j \leq t$; then cyclically shift the 1's over one column for the next row, etc. Add the vector G to each row of $T(g)$. This gives the g integer allocations to be mixed equally to give $\mathcal{B}_o(g)$. Thus,

$$b_{ij} = g - j + t_{ij} \text{ for } j = 1, 2, \ldots, g.$$

For example, if B = 13, then g = 5, and Blue mixes equally over a game matrix:

5	4	3	1	0
4	4	3	2	0
4	3	3	2	1
5	3	2	2	1
5	4	2	1	1

Later we shall show why Blue must average over closest integers to B_i for each i.

4. OPTIMAL ATTACK STRATEGY

Case 1. $R \geq g$.

Red mixes equally over each of the g pure strategies $\mathcal{R}_i$ where

$$\mathcal{R}_i = \{1, 1, \ldots, 1, R - i + 1, 0, \ldots, 0\},$$
$$i = 1, 2, \ldots, g.$$

Thus, he repeatedly "bluffs" (i.e., he sends just one bomber rather than all his remaining planes) until the i-th group, when he sends all his remaining planes. He chooses the integer i uniformly over its range from 1 to g.

Note that Red needs to know only g, not B. In fact, $g = 5$ for $11 \leq B \leq 15$, so he plays the same strategy over this range for B.

Case 2. $R \leq g - 1$.

Here the payoff $M(\mathcal{B}_0, \mathcal{R}) = R$ no matter what strategy Red chooses. Hence all strategies are optimal for Red.

5. VERIFICATION OF OPTIMALITY

THEOREM 1. *The game value* V *is given by*

(5) $$V = \min(B_1, R)$$

where B_1 *is the expected allocation against the first Red group as defined by* $\beta_o(g)$ *and* R *is the total Red force.*

Some lemmas follow.

LEMMA 1. *For* b, x, y, *any nonnegative numbers*

(6) $$\min(b + 1, x + 1) + \min(b, y) \geq \min(b + 1, 1) + \min(b, x + y).$$

Since (6) can be written as

(7) $$\min(b, x) + \min(b, y) \geq \min(b, x + y),$$

it follows that

If $b \leq x$, $b \leq y$, (7) becomes $b + b \geq b$.

If $x < b \leq y$, (7) becomes $x + b \geq b \geq \min(b, x + y)$.

If $x < b$, $y < b$, (7) becomes $x + y \geq \min(b, x + y)$.

LEMMA 2. *For* x, u, y, v, *any nonnegative numbers,*

(8) $$\min(x, u), + \min(y, v) \leq \min(x + y, u + v).$$

This follows from the fact that the left-hand side is less than or equal to x + y and also less than or equal to u + v.

LEMMA 3. *If* b *and* r *are positive integers, and* $b \leq \bar{b} < b + 1$ *with* $\bar{b} = b + \frac{t}{g}$ *where* $0 \leq t < g$, *then*

(9) $\sum_{1}^{t} \min(b + 1, r) + \sum_{1}^{g-t} \min(b, r) = g \min(\bar{b}, r).$

<u>Case 1.</u> If $r \geq b + 1$, (9) becomes

$t(b + 1) + (g - t)b = gb + t = g\bar{b} = g \min(\bar{b}, r).$

<u>Case 2.</u> If $r \leq b$, (9) becomes

$tr + (g - t) r = gr = g \min(\bar{b}, r).$

This shows that if Blue mixes over the two nearest integers to $\bar{b}$, then this gives the payoff $\min(\bar{b}, r)$. On the other hand, if Blue does not average over b and b + 1 to get $\bar{b}$, the payoff to Blue may be smaller.

For example, if $b_1 < r < b_2$ and are all integers, note that

(10) $$\min(b_1, r) + \min(b_2, r) < 2 \min\left(\frac{b_1 + b_2}{2}, r\right),$$

since then $b_1 + r < 2r$, and $b_1 + r < b_1 + b_2$. This explains the necessity of the detailed construction of $\mathcal{B}_0(g)$.

To illustrate, consider the above example for B = 13. Adjust the first and second columns in the first and fourth row to give an expected assignment of B_1 as before, but now the values are 6, 5, 4 to average $4.6 = B_1$. The game matrix is now

6	3	3	1	0
4	4	3	2	0
4	3	3	2	1
4	4	2	2	1
5	4	2	1	1

Then the payoff against $\mathcal{B}_o$ (5) for R = 5 is 4.56, rather than 4.6 when Blue mixes over nearest integers to B_i.

PROOF OF THEOREM 1. If Blue plays $\mathcal{B}_o(g) = \mathcal{B}_o$ and Red plays any pure strategy $\mathcal{R}$, then we show that

$$\text{(11)} \qquad M(\mathcal{B}_o, \mathcal{R}) \geq \min(B_1, R).$$

Let Red establish $p \leq R$ with his strategy

$$\mathcal{R}(p) = \{r_1, r_2, \ldots, r_p\},$$

and let the bluffing strategy for p groups be

$$\mathcal{R}^*(p) = \{1, 1, \ldots, 1, R - p + 1\}.$$

We first show that

$$M(\mathcal{B}_o, \mathcal{R}(p)) \geq M(\mathcal{B}_o, \mathcal{R}^*(p)).$$

By applying Lemma 1 repeatedly, shifting any excess Red allocation to the next following group shows that $\mathcal{R}(p)$ can be adjusted to $\mathcal{R}^*(p)$ without any loss to Red (i.e., bluffing with just one bomber is optimal).

Suppose $p \leq g$; then $M(\mathcal{B}_o, \mathcal{R}^*(p)) =$

$\sum_{i=1}^{g} \min(B_i, r_i) = p - 1 + \min(B_p, r_p) =$

$p - 1 + \min(B_1 - (p - 1), R - (p - 1)) =$

$\min(B_1, R)$.

If $p > g$, then $R > g > B_1$ and $M(\mathcal{B}_o, \mathcal{R}^*(1)) =$

$g - 1 + \min(B_g, 1) =$

$\min(g - 1 + B_g, g) = \min(B_1, g) =$

$B_1 = \min(B_1, R)$.

Thus any Red bluffing strategy $\mathcal{R}^*(p)$ has the same payoff, i.e., $\min(B_1, R)$, against the optimal Blue strategy $\mathcal{B}_o$, and Eq. (11) is verified.

Next we show for any Blue strategy $\mathcal{B}$ against $\mathcal{R}_o(g)$ that

(12) $$M(\mathcal{B}, \mathcal{R}_o(g)) \leq \min(B_1, R).$$

Consider two cases.

Case 1. If $R \leq g - 1 \leq B_1$, then $M(\mathcal{B}, \mathcal{R}) \leq R = \min(B_1, R)$.

Case 2. If $\mathcal{R} \geq g > B_1$, then let

$$\mathcal{B}(q) = \{b_1, b_2, \ldots, b_q\}, \quad \text{where } \sum_{i=1}^{q} b_i = B$$

and the b_i are integers. Since Blue is playing against $\mathcal{R}_o(g)$, clearly he chooses $q \leq g$ without loss in payoff.

Recall that $\mathcal{R}_o(g) = \mathcal{R}_o$ is an equal mixture of g bluffing strategies so that

$$
\begin{array}{rl}
 & gM(\beta(q), \mathcal{R}_o) \\
 & = \min(b_1, R) \\
 & + \min(b_1, 1) + \min(b_2, R - 1) \\
 & + \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & + \min(b_1, 1) + \ldots + \min(b_{q-1}, 1) + \min(b_q, R - (q - 1)) \\
 & + \min(b_1, 1) + \ldots + \min(b_{q-1}, 1) + \min(b_q, 1) \\
\begin{array}{c} g-q \\ \text{rows} \end{array} & \left\{ \begin{array}{l} \cdot \\ \cdot \\ \cdot \end{array} \right. \\
 & + \min(b_1, 1) + \ldots + \min(b_{q-1}, 1) + \min(b_q, 1) \\
 & = \sum_1^q \min(b_i, \mathcal{R} - i + 1) + (1 + 2 + \ldots + g - 1) \\
 & - (1 + \ldots + g - q - 1)
\end{array}
$$

or

$$
(13) \qquad gM(\beta(q), \mathcal{R}_o) = \sum_1^q \min(b_i, R - i + 1) + \frac{(g - 1)g}{2} - \frac{(g - q - 1)(g - q)}{2}.
$$

By using Lemma 2 repeatedly,

$$
\begin{aligned}
\sum_1^q \min(b_i, R - i + 1) &\leq \min\left(B, \sum_1^q (R - i + 1)\right) \\
&\leq \min\left(B, \sum_1^g (R - i + 1)\right).
\end{aligned}
$$

Thus for $q \leq g$,

$$\begin{aligned} gM(\mathcal{B}(q), \mathcal{R}_o(g)) &\leq gM(\mathcal{B}(g), \mathcal{R}_o(g)) \\ &\leq \frac{(g-1)g}{2} + \min\left(B, \sum_{1}^{g} (R - i + 1)\right) \\ &\leq \frac{(g-1)g}{2} + \min\left(B, gR - \frac{(g-1)g}{2}\right) \\ &\leq \min\left(B + \frac{(g-1)g}{2}, gR\right) \\ &= \min(gB_1, gR) \\ &= g \min(B_1, R). \end{aligned}$$

Thus $M(\mathcal{B}(q), \mathcal{R}_o(g)) \leq \min(B_1, R)$, and the value of the game is

$$V = \min(B_1, R).$$

An implication of the results of this game would be that the inability of the defense to count the number of bombers in each group would be very costly to him. In the case where Blue has 10 missiles against 10 bombers, if Blue can count the size of each group, he kills all 10 bombers, but if Red countermeasures could prevent that counting, then Blue can expect to kill only 4.

6. GENERAL DEFENSE SALVOS ($k > 1$)

Note that in the case when $k = 1$, with Blue having just one chance to hit each Red group, any sequential information Blue learns about the number of Red survivors cannot help him as he plays independent of Red's strength.

However, when Blue has $k > 1$ chances to use shoot-look-shoot tactics on each Red group, the game becomes much more complicated. No general solution has been found, but some small examples have been solved to illustrate the nature of the optimal strategies and the effect of k on the value of the game. These examples indicate that Red will bluff a number of times with up to k bombers in each group, and then send all his remaining force in the final group. Blue gets sequential information while Red does not. Blue no longer plays independently of Red's force.

For $k = 2$, the game matrix is expressed as a sequential game in terms of first move choices as far as the analysis allows for such an approach. The game value $V(B, R)$ is tabulated for the following values of R, and B:

B \ R	2	3	4	5
2	2	2	2	2
3	2	$2\frac{1}{3}$	$2\frac{2}{3}$	$2\frac{2}{3}$
4	2	3	$3\frac{2}{5}$	$3\frac{2}{5}$
5	2	3	$3\frac{4}{5}$	

For min(B, R) = 2, it is clear that V(B, R) = 2 since Blue can use shoot-look-shoot tactics or hold for the next group depending on his observation.

For B = 3, R = 3, the game matrix described in terms of the first group allocation is as follows: Red chooses to send 1, 2 or 3 in the first group and Blue will allocate 1 + 1, 1 + 2, 2 + 1 to be sent against the first group, i.e., 1 + 2 means that Blue sends 1, then looks, and if there are survivors in the first group, Blue sends 2. Note that he will partition his allocation into k parts, so that the strategy 3 + 0 is out. Then the game matrix is

Red / Blue	3	2	1
1+1	1+1+M(1, 0) = 2	1+1+M(1, 1) = 3	1+M(2, 2) = 3
1+2	1+2+M(0, 0) = 3	1+1+M(0, 1) = 2	1+M(2, 2) = 3
2+1	2+1+M(0, 0) = 3	2+M(1, 1) = 3	1+M(1, 2) = 2

so that each side plays uniformly on his 3 strategies, and V(3, 3) = $2\frac{2}{3}$. Next, for B = 3 and R > 3 we have

Red / Blue	R	$R-1 \geq R_1 \geq 3$	2	1
1+1	2	$2+M(1, R-R_1) = 3$	2+M(1, R−2) = 3	1+M(2, R−1) = 3
1+2	3	$3+M(0, R-R_1) = 3$	2+M(0, R−2) = 2	1+M(2, R−1) = 3
2+1	3	$3+M(0, R-R_1) = 3$	2+M(1, R−2) = 3	1+M(1, R−1) = 2

It is clear that Red should send 1, 2 or R on his first group. Thus $V(3, R) = V(3, 3) = 2\frac{2}{3}$. Also, for $B \geq 4$, $V(B, 3) = 3$ if Blue plays 1 + 2.

Next, for $B = 4$, $R = 4$, we have

Blue \ Red	4	3	2	1
1+1	2	2+M(2, 1) = 3	2+M(2, 2) = 4	1+M(3, 3)
1+2	3	3+M(1, 1) = 4	2+M(1, 2) = 3	1+M(3, 3)
1+3	4	4+M(0, 1) = 3	2+M(0, 2) = 2	1+M(3, 3)
2+1	3	3+M(1, 1) = 4	2+M(2, 2) = 4	1+M(2, 3) = 3
2+2	4	3+M(0, 1) = 3	2+M(2, 2) = 4	1+M(2, 3) = 3
3+1	4	3+M(1, 1) = 4	2+M(1, 2) = 3	1+M(1, 3) = 2

Here we pause a moment to consider the situation for $M(3, 3)$. Blue knows $B = R = 3$ but Red knows only that $R = 3$ and $B \leq 3$. If $B \leq 2$, then the payoff $M(B, 3)$ will be B regardless of what Red does. Therefore, Red loses nothing by assuming that $B = 3$ and plays accordingly with $M(3, 3) = 2\frac{2}{3}$. The above matrix is now

2	3	4	$3\frac{2}{3}$
3	4	3	$3\frac{2}{3}$
4	3	2	$3\frac{2}{3}$
3	4	4	3
4	3	4	3
4	4	3	2

Red's optimal strategy is (.2, 0, .2, .6), while Blue's optimal strategy is (.1, .4, .1, 0, .4, 0), giving $V(4, 4) = 3.4$. Note that once again Red plays R, 2 or 1 on his first move. Similarly, if $B = 4$, $R \geq 5$, the matrix is written below with the simplified notation (b, r) for M(b, r).

Blue \ Red	R	$R > R_1 > 3$	3	2	1
1+1	2	$2+(2, R-R_1) \geq 3$	$2+(2,x) = 4$	$2+(2,y) = 4$	$1+M(3,z)$
1+2	3	$3+(1, R-R_1) = 4$	$3+(1,x) = 4$	$2+(1,y) = 3$	$1+M(3,z)$
1+3	4	$3+(0, R-R_1) = 3$	$3+(0,x) = 3$	$2+(0,y) = 2$	$1+M(3,z)$
2+1	3	$3+(1, R-R_1) = 4$	$3+(1,x) = 4$	$2+(2,y) = 4$	$1+(2,z) = 3$
2+2	4	$4+(0, R-R_1) = 4$	$3+(0,x) = 3$	$2+(2,y) = 4$	$1+(2,z) = 3$
3+1	4	$4+(0, R-R_1) = 4$	$3+(1,x) = 4$	$2+(1,y) = 3$	$1+(1,z) = 2$

where $x = R - 3$, $y = R - 2$, $z = R - 1$.

As before, consider the situation after Red sends out one bomber in the first group, and Blue sends one missile against it. Here Red knows only that $B \leq 3$. If $B \leq 2$, then the payoff $M(B, R-1) = B$ regardless of what Red does, so Red loses nothing when he assumes that $B = 3$ and plays accordingly. From this we have $M(3, R-1) = V(3, R-1) = V(3, 3) = 2\frac{2}{3}$. The game matrix now becomes

	.2			.2	.6
.1	2	≥ 3	4	4	$3\frac{2}{3}$
.4	3	4	4	3	$3\frac{2}{3}$
.1	4	3	3	2	$3\frac{2}{3}$
	3	4	4	4	3
.4	4	4	3	4	3
	4	4	4	3	2

with $V(4, R) = V(4, 4) = 3.4$. Thus, once again Red plays R, or 2, or 1 on the first move, and Blue plays the same for $R > B$ as for $R = B$. Also $V(B, R) = V(B, B)$ for $R > B$. These results could be conjectured in general.

Next, for $B = 5$ we have $V(5, 3) \geq V(4, 3)$, so $V(5, 3) = 3$. For $R = 4$, Blue can limit the total first group allocation to a value $\leq R$. Thus, there are six first move strategies for Blue.

$B = 5, R = 4$

Red / Blue	4	3	2	1
1+1	2	2+(3, 1) = 3	2+(3, 2) = 4	1+(4, 3) = 4
1+2	3	3+(2, 1) = 4	2+(2, 2) = 4	1+(4, 3) = 4
1+3	4	3+(1, 1) = 4	2+(1, 2) = 3	1+(4, 3) = 4
2+1	3	3+(2, 1) = 4	2+(3, 2) = 4	1+(3, 3) = $3\frac{2}{3}$
2+2	4	3+(1, 1) = 4	2+(3, 2) = 4	1+(3, 3) = $3\frac{2}{3}$
3+1	4	3+(2, 1) = 4	2+(2, 2) = 4	1+(2, 3) = 3

Here, after Red has played 1 in his first group he knows only that $B \leq 4$. But if $B = 4$, $M(B, 3) = 3$ regardless of what Red plays. Also, if $B \leq 2$, $M(B, 3) = B$ regardless of what Red plays. Thus, Red can assume $B = 3$ and play the Red strategy which was optimal for that case. This gives $M(3, 3) = V(3, 3) = 2\frac{2}{3}$.

Next, the first row is dominated by the second, the fourth by the fifth and the sixth by the fifth. Thus, the second column is out and the reduced matrix is

3	4	4
4	3	4
4	4	$3\frac{2}{3}$

Red plays $\{\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\}$, while Blue plays $\{\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\}$ and the game value $V(5, 4) = \frac{19}{5}$. Note that once more Red plays R, 2, or 1 on the first move.

The final case to be discussed is $B = 5$, $R = 5$. Here the game matrix is

Red \ Blue	5	4	3	2	1
1+1	2	2+(3, 1) = 3	2+(3, 2) = 4	2+(3, 3) = 4$\frac{2}{3}$	1+(4, 4)
1+2	3	3+(2, 1) = 4	3+(2, 2) = 5	2+(2, 3) = 4	1+(4, 4)
1+3	4	4+(1, 1) = 5	3+(1, 2) = 4	2+(1, 3) = 3	1+(4, 4)
1+4	5	4+(0, 1) = 4	3+(0, 2) = 3	2+(0, 3) = 2	1+(4, 4)
2+1	3	3+(2, 1) = 4	3+(2, 2) = 5	2+(3, 3) = 4$\frac{2}{3}$	1+(3, 4)
2+2	4	4+(1, 1) = 5	3+(1, 2) = 4	2+(3, 3) = 4$\frac{2}{3}$	1+(3, 4)
2+3	5	4+(0, 1) = 4	3+(0, 2) = 3	2+(3, 3) = 4$\frac{2}{3}$	1+(3, 4)
3+1	4	4+(1, 1) = 5	3+(2, 2) = 5	2+(2, 3) = 4	1+(2, 4) = 3
3+1	5	4+(0, 1) = 4	3+(2, 2) = 5	2+(2, 3) = 4	1+(2, 4) = 3
4+1	5	4+(1, 1) = 5	3+(1, 2) = 4	2+(1, 3) = 3	1+(1, 4) = 2

After Red plays 2 on his first move, he knows only that $B \leq 3$. If $B \leq 2$, the payoff $M(B, 3) = B$ regardless of what Red does. So Red can assume $B = 3$ and play as in the (3, 3) case. Thus $M(3, 3) = 2\frac{2}{3}$. However, after Red has played 1 on his first move, he knows only that $B \leq 4$. If $B \leq 2$, then the payoff is $M(B, 4) = B$ regardless of what Red does. But now Red can assume only that $B = 3$ or 4. Since Red's optimal strategy against $B = 3$ is not the same as that against $B = 4$, the method of expressing the game matrix in terms of sequential subgames breaks down at this point and thus one is forced to work out an expanded game matrix.

Nevertheless, the special argument which was used repeatedly in the above cases which have been solved in this paper has proved to be very useful here and also in other penetration models which the author has studied. The

principle is now stated formally.

<u>OPTIMAL PRINCIPLE.</u> <u>In a multimove game suppose player Red is uncertain of the exact size of player Blue's strength, which ranges over a finite set of integers. If the payoff from that point on in the game is independent of Red's strategy for all but one value of Blue's strength, Red can then assume that one value for Blue and play optimally against that value without loss to himself</u>.

Finally, it should be noted that the character of the strategies and the game is further altered by the introduction of a probability of kill < 1.

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10. ABSTRACT

An investigation of the question of how valuable it is to the offense to have countermeasures to prevent the defense from counting the number of units in the force of an air attack. An attacker has a force of bombers or missiles that he is trying to get past a defense position defended by homing missiles; he can split up his force into one or more groups of different weights. The attacker's countermeasures prevent the defense from counting the number of units in each group. The defense can salvo up to k times at a given group with shoot-look-shoot tactics. Total attack and defense size are known to both sides. A game theoretic solution is presented for the case of k equals 1, with one defense salvo per group. The case of k greater than 1 is discussed and solved for several small examples.

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11. KEY WORDS

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